

Some remarks on singular solutions of nonlinear elliptic equations. III: viscosity solutions, including parabolic operators

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0 Introduction

One of the main results in [3], theorem 1.1, is a strong maximum principle for a singular supersolution u in a domain Ω in \mathbb{R}^n lying above a C^2 solution v , i.e. with $u \geq v$. Recently

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we observed that under the conditions in the theorem, indeed under weaker conditions and also in theorems 1.2 and 1.3 there, the function u satisfies in all of Ω

$$F(x, u, \nabla u, \nabla^2 u) \leq F(x, v, \nabla v, \nabla^2 v) \quad \text{in viscosity sense.} \quad (1)$$

(Note that in this paper $F(x, u, \nabla u, \nabla^2 u)$ amounts to $F(x, u, \nabla u, -\nabla^2 u)$ in [3] — a change of notation.)

Furthermore, we found that the strong maximum principle holds for functions u which are lower-semi-continuous (LSC) and satisfy (1) in Ω .

Throughout, the nonlinear operator $F(x, s, p, M)$ is assumed to be elliptic for all values of the arguments, and C^1 in (s, p, M) , but not uniformly elliptic; nor are $|F_s|$, $|F_s|$ uniformly bounded.

In section 1 we prove that the singular functions u satisfying the modified condition satisfy (1).

We would like to point out some new ingredients in the arguments.

In theorem 1.1 in [3] we considered a function u with possible singularity at a point, say the origin. We used a condition that for any $r > 0$ small,

$$\inf_{0 < |x| \leq r} (u + \text{any linear function}) \text{ occurs on } \{|x| = r\}. \quad (2)$$

(This condition is related to the notation of superaffine, as described in [3].)

In this paper, in section 1 we start by showing that under a new weaker condition than (2), a viscosity supersolution on $0 < |x| < r$ of (1) becomes a viscosity supersolution in $|x| < r$. Namely, we introduce a class of functions which, to our knowledge, is new and which may prove useful in further work: lowerconical functions. A function u is lowerconical at a point $\bar{x} \in \Omega$, if for any $\eta \in C^\infty(\Omega)$, and for any $\epsilon > 0$,

$$\inf_{x \in \Omega} \left((u + \eta)(x) - (u + \eta)(\bar{x}) - \epsilon|x - \bar{x}| \right) < 0.$$

This is formulated more precisely in Definition 1.1. Theorem 1.4 generalizes Theorem 1.1 to viscosity supersolutions outside a closed manifold,

The notion lowerconical makes sense also on a Riemannian manifold. As we pointed out in Remark 1.2, this condition is almost necessary for a viscosity supersolution.

We use another ingredient: a sharpening of the Hopf Lemma. It is used in our proof that a viscosity supersolution in a punctured region is also one in the whole region. The sharp form of the Hopf Lemma, Lemma 1.1, refers to a linear second order uniformly elliptic operator L in a bounded domain Ω with C^2 boundary, and to a function $u \geq 0$ in Ω with

$$u \geq 1 \quad \text{in a ball } B_\delta \text{ in } \Omega.$$

Lemma 1.1 states that there exist $\bar{\epsilon}, \bar{\mu} > 0$ depending only on n, δ, Ω , and the ellipticity constants, such that if

$$Lu \leq \bar{\epsilon} \text{ in } \Omega \text{ in viscosity sense,}$$

then

$$u(x) \geq \bar{\mu} \operatorname{dist}(x, \partial\Omega).$$

Lemma 1.1 follows from the special case, Lemma 1.2, where Ω is a ball. It is a bit surprising that we actually need this form of the Hopf Lemma. In section 1 we give an “elliptic” proof of it. Lemma 1.2 is also an immediate corollary of a corresponding sharp form of the Hopf Lemma for parabolic operators, see Theorem 5.1 in section 5. Lemma 1.2 follows from it by considering u independent of t . The analogue of Lemma 1.1, for parabolic operators, is given in Theorem 5.2.

Here is an outline of the other sections. First, section 2 is concerned with the maximum principle for LSC viscosity supersolutions u of (1) in Ω , where $v \in C^2$, in case $u \geq v$ on $\partial\Omega$.

Question. Does the maximum principle hold, i.e.,

$$u \geq v \text{ in } \Omega,$$

if, say, Ω is a small ball?

In general, no, not even for smooth u in case F is not uniformly elliptic (see Example 2.1). But in Theorem 2.1, we prove the maximum principle if

$$F_u \leq 0$$

— under a rather mild ellipticity condition on F .

Using a very different kind of argument, in section 1.2, we also prove that the maximum principle holds, without assuming $F_u \leq 0$, in case u satisfies some linear elliptic inequalities.

In section 3 we prove the strong maximum principle for u , LSC, satisfying (1) in viscosity sense in Ω . We also present an extension of the Hopf Lemma for viscosity supersolutions; uniform ellipticity is never required.

In section 4 we extend the strong maximum principle and the Hopf Lemma to viscosity supersolutions of nonlinear parabolic operators. Section 4 is self-contained and may be read independently of the others. We thank H. Matano for suggesting that we consider the problem.

1 Removable singularities for viscosity solutions

1.1 A sufficient condition

Let $F \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n})$, where $\mathcal{S}^{n \times n}$ denotes the set of $n \times n$ real symmetric matrices and Ω is a domain (bounded connected open set) in the n -dimensional Euclidean space \mathbb{R}^n . Throughout the paper we use $B_r(x)$ to denote a ball of radius r and centered at x , and use B_r to denote $B_r(0)$. We use $LSC(\Omega)$ and $USC(\Omega)$ to denote respectively the set of lower-semicontinuous and upper-semicontinuous functions.

Definition 1.1 *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in LSC(\Omega)$ satisfying*

$$\inf_{\Omega} u > -\infty. \quad (3)$$

We say that u is lowerconical at a point $\bar{x} \in \Omega$, if for any $\eta \in C^\infty(\Omega)$, and for any $\epsilon > 0$,

$$\inf_{x \in \Omega} \left((u + \eta)(x) - (u + \eta)(\bar{x}) - \epsilon|x - \bar{x}| \right) < 0.$$

We say that u is upperconical at $\bar{x} \in \Omega$, if $-u$ is lowerconical at \bar{x} .

We say that u is lowerconical on a subset of E of Ω , if for any $\eta \in C^\infty(\Omega)$, and for any $\bar{x} \in E$ and any $\epsilon > 0$,

$$\inf_{x \in \Omega} \left((u + \eta)(x) - (u + \eta)(\bar{x}) - \epsilon \text{dist}(x, E) \right) < 0, \quad (4)$$

where $\text{dist}(x, E)$ denotes the distance of x to E . Similarly we say that u is upperconical on E if $-u$ is lowerconical on E .

Note that for a smooth submanifold E of dimension $1 \leq k \leq n-1$, $u(x) := \text{dist}(x, E)$ is lowerconical at every point $\bar{x} \in E$, but it is not lowerconical on E .

Remark 1.1 *If u is differentiable at \bar{x} , then u is both lowerconical and upperconical at \bar{x} . In fact, if for some C^1 curve $\gamma(t)$ satisfying $\gamma(0) = \bar{x}$, $u(\gamma(t))$ is differentiable at 0, then u is both lowerconical and upperconical at \bar{x} . On the other hand, $u(x) = |x|$, a Lipschitz function, is not lowerconical at 0. It is easy to see that if $\liminf_{x \rightarrow \bar{x}} u(x) > u(\bar{x})$, then u is not lowerconical at \bar{x} . Also, $u(x) = \sin(1/|x|)$ for $x \neq 0$, $u(0) = -1$, is both lowerconical and upperconical at 0, but is not even continuous.*

Theorem 1.1 For $n \geq 1$, let Ω be a domain in \mathbb{R}^n , $\bar{x} \in \Omega$, and let $F \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n})$. Assume that $u \in LSC(\Omega)$ is lowerconical at $\{\bar{x}\}$ and satisfies, for some $f \in USC(\Omega)$,

$$F(x, u, \nabla u, \nabla^2 u) \leq f(x), \quad \text{in } \Omega \setminus \{\bar{x}\} \text{ in viscosity sense.} \quad (5)$$

Then

$$F(x, u, \nabla u, \nabla^2 u) \leq f(x), \quad \text{in } \Omega \text{ in viscosity sense.} \quad (6)$$

Remark 1.2 There is a kind of converse. Namely, if F is further assumed to satisfy

$$\limsup_{a \rightarrow \infty} \inf_{x \in \Omega, |(s,p)| \leq \beta} F(x, s, p, aI) = \infty, \quad \forall \beta > 0, \quad (7)$$

and f is further assumed to satisfy $\sup_{\Omega} f < \infty$, then if $u \in LSC(\Omega)$ and satisfies (3) and (6), necessarily u is lowerconical at every point of Ω . On the other hand, any u satisfies $-e^{-\Delta u} \leq 0$. This operator does not satisfy (7). Condition (7) is clearly satisfied by uniformly elliptic operators.

To see the above, suppose that u is not lowerconical at some point, say 0, in Ω , then for some $\epsilon \in (0, 1)$ and some $\eta \in C^\infty(\Omega)$,

$$(u + \eta)(x) - (u + \eta)(0) - \epsilon|x| \geq 0, \quad \text{in } \Omega.$$

So for some constant $\delta \in (0, 1)$,

$$u(x) \geq u(0) - \nabla \eta(0) \cdot x + \frac{\epsilon}{2}|x|, \quad \forall 0 < |x| < \delta.$$

For $a > 1/(4\delta)$,

$$u(x) > \varphi_a(x) := u(0) - \nabla \eta(0) \cdot x + a|x|^2, \quad \forall |x| = \epsilon/(4a),$$

$$\varphi_a(x) \leq u(0) + |\nabla \eta(0)| + \epsilon/(16a) \leq u(0) + |\nabla \eta(0)| + 1, \quad |x| \leq \epsilon/(4a).$$

Move φ_a down, and then up to position $\varphi_a - \bar{b}$, $\bar{b} \geq 0$, so that its graph first touches that of u from below, at some point \bar{x} , $|\bar{x}| < \epsilon/(4a)$. More precisely, let

$$\bar{b} = \sup\{b \mid u(x) \geq (\varphi_a - b)(x), \forall |x| \leq \epsilon/(4a)\}.$$

Clearly, $\bar{b} \geq 1 + u(0) + |\nabla \eta(0)| - \inf_{\Omega} u$. On the other hand, since $u(0) = \varphi_a(0)$ and $u(x) > \varphi_a(x)$ for all $|x| = \epsilon/(4a)$, we infer that $\bar{b} \geq 0$, and for some $|\bar{x}| < \epsilon/(4a)$, $u(\bar{x}) = (\varphi_a - \bar{b})(\bar{x})$ and $u(x) \geq (\varphi_a - \bar{b})(x)$ for all $|x| \leq \epsilon/(4a)$. By (6),

$$F(\bar{x}, (\varphi_a - \bar{b})(\bar{x}), \nabla(\varphi_a - \bar{b})(\bar{x}), \nabla^2(\varphi_a - \bar{b})(\bar{x})) \leq f(\bar{x}).$$

It is easy to see that $|(\varphi_a - \bar{b})(\bar{x})|$ and $|\nabla(\varphi_a - \bar{b})(\bar{x})|$ are bounded by some constant independent of a . Sending a to ∞ in the above, we arrive at a contradiction in view of (7).

The following example shows that the assumption on u in Theorem 1.1 is essentially optimal.

Example 1.1 *Let $u(x) = |x|$. Then*

$$F(x, u, \nabla u, \nabla^2 u) := -e^{-\Delta u} + 1 - |\nabla u|^2 \leq 0, \quad \text{in } B_1 \setminus \{0\}.$$

But the inequality does not hold in B_1 in viscosity sense, as easily seen by taking $\varphi(x) = |x|^2$ as a test function.

Proof of Theorem 1.1. We may assume that $\bar{x} = 0$. Let $\varphi \in C^2(\Omega)$ satisfy $(u - \varphi)(0) = 0$, $u - \varphi \geq 0$ in Ω . For any $0 < \delta < \text{dist}(0, \partial\Omega)/9$, we consider

$$\varphi_{\delta}(x) := \varphi(x) - \frac{\delta}{2}|x|^2. \quad (8)$$

Clearly,

$$u(x) > \varphi_{\delta}(x), \quad x \in \Omega \setminus \{0\}, \quad (9)$$

$$u(x) \geq \varphi_{\delta}(x) + \frac{1}{2}\delta^3, \quad x \in \Omega, \quad |x| \geq \delta. \quad (10)$$

Since $u \in LSC(\Omega)$ is lowerconical, we have, for large i ,

$$\liminf_{x \rightarrow 0} \left((u - \varphi_{\delta})(x) - (u - \varphi_{\delta})(0) - \frac{1}{i}|x| \right) \geq 0,$$

and there exists $\{x_i\} \subset \Omega \setminus \{0\}$ such that

$$\begin{aligned} & (u - \varphi_{\delta})(x_i) - \frac{1}{i}|x_i| \\ &= (u - \varphi_{\delta})(x_i) - (u - \varphi_{\delta})(0) - \frac{1}{i}|x_i| \\ &= \inf_{x \in \Omega} \left((u - \varphi_{\delta})(x) - (u - \varphi_{\delta})(0) - \frac{1}{i}|x| \right) < 0. \end{aligned} \quad (11)$$

Claim that

$$\lim_{i \rightarrow \infty} x_i = 0. \quad (12)$$

Indeed, let $x_i \rightarrow \hat{x}$ along a subsequence, still denoted as $\{x_i\}$. Then, after sending i to infinity in (11), we have

$$(u - \varphi_\delta)(\hat{x}) \leq 0,$$

which implies $\hat{x} = 0$ in view of (9) and (10). We have proved (12). □

Let

$$\varphi_\delta^{(i)}(x) := \varphi_\delta(x) + \frac{1}{\sqrt{i}} \frac{x_i}{|x_i|} \cdot x.$$

We have, in view of (11) and (12), that

$$(u - \varphi_\delta^{(i)})(x_i) = (u - \varphi_\delta)(x_i) - \frac{1}{\sqrt{i}} |x_i| < \left(\frac{1}{i} - \frac{1}{\sqrt{i}}\right) |x_i| < 0.$$

Since

$$(u - \varphi_\delta^{(i)})(0) = 0,$$

and, in view of (9) and (10),

$$(u - \varphi_\delta^{(i)})(x) \geq \frac{1}{2} \delta^3 + O\left(\frac{1}{\sqrt{i}}\right) > 0, \quad \forall |x| \geq \delta, \text{ for large } i,$$

there exists \tilde{x}_i , $0 < |\tilde{x}_i| < \delta$, such that

$$(u - \varphi_\delta^{(i)})(\tilde{x}_i) = \min_{0 < |x| < \delta} (u - \varphi_\delta^{(i)})(x) < 0.$$

Namely,

$$\psi_\delta^{(i)}(x) := \varphi_\delta^{(i)}(x) + (u - \varphi_\delta^{(i)})(\tilde{x}_i)$$

satisfies $\psi_\delta^{(i)}(\tilde{x}_i) = u(\tilde{x}_i)$ and $u \geq \psi_\delta^{(i)}$ near \tilde{x}_i .

Similar to (12), we have

$$\lim_{i \rightarrow \infty} \tilde{x}_i = 0.$$

Thus, by (5),

$$F(\tilde{x}_i, \psi_\delta^{(i)}(\tilde{x}_i), \nabla \psi_\delta^{(i)}(\tilde{x}_i), \nabla^2 \psi_\delta^{(i)}(\tilde{x}_i)) \leq f(\tilde{x}_i).$$

Sending i to ∞ in the above leads to

$$F(0, \varphi_\delta(0), \nabla \varphi_\delta(0), \nabla^2 \varphi_\delta(0)) \leq f(0).$$

Sending δ to 0 in the above leads to

$$F(0, \varphi(0), \nabla \varphi(0), \nabla^2 \varphi(0)) \leq f(0).$$

Theorem 1.1 is proved. □

1.2 Another sufficient condition for removable singularity

Let $(a_{ij}(x))$, $b_i(x)$ and $c(x)$ be $L^\infty(\Omega)$ functions satisfying, for some positive constants λ and Λ ,

$$|a_{ij}(x)| + |b_i(x)| + |c(x)| \leq \Lambda, \quad a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n. \quad (13)$$

In the rest of this section we assume that F is a degenerate elliptic operator:

$$F(x, s, p, M + N) \geq F(x, s, p, M), \quad \forall (x, s, p, M) \in \mathcal{N}, N \in \mathcal{S}_+^{n \times n}, \quad (14)$$

where $\mathcal{S}_+^{n \times n} \subset \mathcal{S}^{n \times n}$ denotes the set of positive definite matrices.

Theorem 1.2 *For $n \geq 1$, let $F \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n})$ satisfy (14), $(a_{ij}(x))$, $b_i(x)$ and $c(x)$ be as above, and let $f \in USC(\Omega)$. Assume that $u \in LSC(\Omega)$ satisfies, for some constant C ,*

$$a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u \leq C, \quad \text{in } \Omega, \text{ in viscosity sense}, \quad (15)$$

and, for some subset E of Ω of zero Lebesgue measure,

$$F(x, u, \nabla u, \nabla^2 u) \leq f(x) \quad \text{in } \Omega \setminus E \text{ in the viscosity sense}. \quad (16)$$

Then

$$F(x, u, \nabla u, \nabla^2 u) \leq f(x) \quad \text{in } \Omega \text{ in the viscosity sense}.$$

Proof of Theorem 1.2. Let $\varphi \in C^2(\Omega)$ satisfy

$$\varphi \leq u, \quad \text{in } \Omega, \quad \text{and } \varphi(\bar{x}) = u(\bar{x}), \text{ for some } \bar{x} \in \Omega.$$

We have only to prove that

$$F(\bar{x}, \varphi(\bar{x}), \nabla \varphi(\bar{x}), \nabla^2 \varphi(\bar{x})) \leq f(\bar{x}). \quad (17)$$

We need only consider $\bar{x} \in E$ and may assume, without loss of generality, that $\bar{x} = 0 \in E$. For any $0 < \delta < \text{dist}(0, \partial\Omega)/9$, let φ_δ be defined in (8). Then (9) and (10) hold.

For $\epsilon \in (0, \delta^3/4)$, let,

$$w_\epsilon(x) \equiv w_\epsilon^{(\delta)}(x) := \begin{cases} \min\{(u - \varphi_\delta)(x) - \epsilon, 0\} & x \in B_\delta, \\ 0 & x \in B_{2\delta} \setminus B_\delta, \end{cases}$$

and

$$\Gamma_{w_\epsilon}(x) := \sup\{a + b \cdot x \mid a \in \mathbb{R}, b \in \mathbb{R}^n, a + b \cdot z \leq w_\epsilon(z) \ \forall z \in B_{2\delta}\}$$

be the convex envelope of w_ϵ on $B_{2\delta} \equiv B_{2\delta}(0)$.

Since $w_\epsilon = 0$ outside B_δ , and $\min_{B_{2\delta}} \Gamma_{w_\epsilon} \leq w_\epsilon(0) = -\epsilon < 0$, the contact set of w_ϵ and Γ_{w_ϵ} satisfies

$$\{x \in B_{2\delta} \mid w_\epsilon(x) = \Gamma_{w_\epsilon}(x)\} \subset \{x \in B_\delta \mid w_\epsilon(x) = (u - \varphi_\delta)(x) - \epsilon < 0\}. \quad (18)$$

We will need

Proposition 1.1 *There exists some positive constants K such that for any point $\bar{x} \in \{x \in B_{2\delta} \mid w_\epsilon(x) = \Gamma_{w_\epsilon}(x)\}$, there exists $\bar{p} \in \mathbb{R}^n$ so that*

$$\Gamma_{w_\epsilon}(x) \leq \Gamma_{w_\epsilon}(\bar{x}) + \bar{p} \cdot (x - \bar{x}) + K|x - \bar{x}|^2, \quad \forall x \in B_{2\delta}.$$

The proof of this proposition will be postponed to the end of the proof of the theorem.

Once Proposition 1.1 is proved, we can apply, as in section 3 of [3], lemma 3.5 of [2] to obtain that $\Gamma_{w_\epsilon} \in C_{loc}^{1,1}(B_{2\delta})$, and then use the Alexandrov-Bakelman-Pucci estimate to obtain

$$\epsilon^n = |\inf_{B_{2\delta}} w_\epsilon|^n \leq \int_{\{w_\epsilon = \Gamma_{w_\epsilon}\}} \det(\nabla^2 \Gamma_{w_\epsilon}).$$

This implies that

$$\text{The Lebesgue measure of } \{w_\epsilon = \Gamma_{w_\epsilon}\} > 0.$$

Since Γ_{w_ϵ} is convex, it is, by the Alexandrov theorem, second order differentiable except on a set of zero Lebesgue measure. We also know that E has zero Lebesgue measure.

Thus we can pick a point $x (= x_\epsilon)$ in $\{w_\epsilon = \Gamma_{w_\epsilon}\} \cap (B_{2\delta} \setminus E)$ where Γ_{w_ϵ} is second order differentiable.

We know from (18) and the definition of Γ_{w_ϵ} that

$$u(x) = \psi_\epsilon(x) := \varphi_\delta(x) + \epsilon + \Gamma_{w_\epsilon}(x), \quad (19)$$

and

$$u \geq \psi_\epsilon = \varphi_\delta + \epsilon + \Gamma_{w_\epsilon}, \quad \text{near } x.$$

Since Γ_{w_ϵ} is second order differentiable at x , $\nabla \Gamma_{w_\epsilon}(x)$ is well defined, and, for any $\mu > 0$, and for z near x ,

$$\begin{aligned} u(z) &\geq (\Gamma_{w_\epsilon} + \varphi_\delta)(x) + \epsilon + \nabla(\Gamma_{w_\epsilon} + \varphi_\delta)(x) \cdot (z - x) \\ &\quad + \frac{1}{2}(z - x)^t \nabla^2(\Gamma_{w_\epsilon} + \varphi_\delta)(x) \cdot (z - x) + o(|z - x|^2) \\ &\geq (\Gamma_{w_\epsilon} + \varphi_\delta)(x) + \epsilon + \nabla(\Gamma_{w_\epsilon} + \varphi_\delta)(x) \cdot (z - x) \\ &\quad + \frac{1}{2}(z - x)^t \nabla^2(\Gamma_{w_\epsilon} + \varphi_\delta)(x) \cdot (z - x) - \frac{\mu}{2}|z - x|^2. \end{aligned}$$

By (16) and the above,

$$F(x, (\Gamma_{w_\epsilon} + \varphi_\delta)(x) + \epsilon, \nabla(\Gamma_{w_\epsilon} + \varphi_\delta)(x), \nabla^2(\Gamma_{w_\epsilon} + \varphi_\delta)(x) - \mu I) \leq f(x).$$

Sending μ to 0 leads to

$$F(x, (\Gamma_{w_\epsilon} + \varphi)(x) + \epsilon, \nabla(\Gamma_{w_\epsilon} + \varphi)(x), \nabla^2(\Gamma_{w_\epsilon} + \varphi)(x)) \leq f(x). \quad (20)$$

Clearly,

$$|\Gamma_{w_\epsilon}(x)| \leq \epsilon, \quad |\nabla \Gamma_{w_\epsilon}(x)| \leq \frac{\epsilon}{\delta}. \quad (21)$$

By the convexity of Γ_{w_ϵ} , $\nabla^2 \Gamma_{w_\epsilon}(x) \geq 0$. We see from (19) and (21) that (recall that $x = x_\epsilon$) $(u - \varphi_\delta)(x_\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. This which implies, in view of (9), (14), (20) and the convexity of Γ_{w_ϵ} , that $x_\epsilon \rightarrow 0$ and

$$\begin{aligned} f(0) &\geq \limsup_{\epsilon \rightarrow 0} F(x, (\Gamma_{w_\epsilon} + \varphi_\delta)(x) + \epsilon, \nabla(\Gamma_{w_\epsilon} + \varphi_\delta)(x), \nabla^2(\Gamma_{w_\epsilon} + \varphi_\delta)(x)) \\ &\geq \limsup_{\epsilon \rightarrow 0} F(x, (\Gamma_{w_\epsilon} + \varphi_\delta)(x) + \epsilon, \nabla(\Gamma_{w_\epsilon} + \varphi_\delta)(x), \nabla^2 \varphi_\delta(x)) \\ &= F(0, \varphi_\delta(0), \nabla \varphi_\delta(0), \nabla^2 \varphi_\delta(0)) \\ &= F(0, \varphi(0), \nabla \varphi(0), \nabla^2 \varphi(0) - \delta I). \end{aligned}$$

Sending δ to 0 in the above leads to

$$F(0, \varphi(0), \nabla \varphi(0), \nabla^2 \varphi(0) \leq f(0).$$

Theorem 1.2 is established — provided Proposition 1.1 holds.

Now we prove Proposition 1.1. Under $\Delta u \leq C$ instead of (13), the above proposition was proved in [3], see lemma 3.1 there. The new ingredient is the following

1.3 A strengthening of the Hopf Lemma

Lemma 1.1 *Let Ω be a domain of \mathbb{R}^n , with C^2 boundary, and let $(a_{ij}(x))$, $b_i(x)$ and $c(x)$ be $L^\infty(\Omega)$ functions satisfying (13) for some positive constants λ and Λ . Let $B \subset \Omega$ be a ball of radius δ . Then there exist some positive constants $\bar{\epsilon}, \bar{\mu} > 0$ which depend only on $n, \lambda, \Lambda, \delta, \Omega$ such that if $u \in LSC(\Omega)$ satisfies*

$$a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u \leq \bar{\epsilon}, \quad \text{in } \Omega, \text{ in viscosity sense,}$$

$$u \geq 0, \quad \text{in } \Omega, \quad \text{and } u \geq 1, \quad \text{on } B.$$

Then

$$u(x) \geq \bar{\mu} \operatorname{dist}(x, \partial\Omega), \quad \text{in } \Omega.$$

We first prove Lemma 1.1 for $\Omega = B_1$, which is stated as

Lemma 1.2 *Let $(a_{ij}(x))$, $b_i(x)$ and $c(x)$ be $L^\infty(\Omega)$ functions satisfying (13) with $\Omega = B_1$ for some positive constants λ and Λ . Then, for any $0 < \delta < 1$, there exist some positive constants $\bar{\epsilon}, \bar{\mu} > 0$ which depend only on $n, \lambda, \Lambda, \delta$, such that if $u \in LSC(\Omega)$ satisfies*

$$Lu := a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u \leq \bar{\epsilon}, \quad \text{in } B_1, \text{ in viscosity sense,} \quad (22)$$

$$u \geq 1, \quad \text{on } B_\delta \subset B_1, \quad (23)$$

and

$$u \geq 0, \quad \text{in } B_1.$$

Then

$$u \geq \bar{\mu}(1 - |x|), \quad \text{on } B_1.$$

Lemma 1.1 then follows by repeated application of this for scaled balls.

Proof of Lemma 1.2. For a large positive constant α to be chosen later, consider the function

$$v(x, t) := \frac{u(x)}{\cos(\alpha t)}, \quad \text{in } B_1 \times (-\beta, \beta), \quad \beta := \frac{\pi}{10\alpha}.$$

In particular, consider v in the ellipsoid

$$E_1 := \{(x, t) \mid |x|^2 + \beta t^2 < 1\}.$$

By (23),

$$v \geq 1, \quad \text{in } E_\delta := \{(x, t) \mid |x|^2 + \beta t^2 < \delta^2\}.$$

By (22),

$$(L + \partial_t^2)u \leq \bar{\epsilon}, \quad \text{in } E_1, \text{ in viscosity sense.}$$

A computation gives

$$(L + \partial_t^2)u = \cos(\alpha t)(Lv + v_{tt}) - 2\alpha \sin(\alpha t)v_t - \alpha^2(\cos(\alpha t))v.$$

Hence

$$\tilde{L}v := a_{ij}v_{ij} + b_i v_i + v_{tt} - 2\alpha \tan(\alpha t)v_t - (\alpha^2 - c)v \leq \frac{\bar{\epsilon}}{\cos(\alpha t)} \leq 2\bar{\epsilon}.$$

We now fix the value of α to be $\sqrt{\Lambda}$. Then,

$$(\alpha^2 - c) \geq 0, \quad \text{in } E_1. \quad (24)$$

In $E_1 \setminus E_\delta$, consider the comparison function

$$h(x, t) := \frac{E - e^{-k}}{D}, \quad E := e^{-k(|x|^2 + \beta t^2)}, \quad D := e^{-k\delta^2} - e^{-k}.$$

Then

$$\begin{aligned} h_i &= -2kx_i \frac{E}{D}, \quad h_t = -2k\beta t \frac{E}{D}, \\ h_{ij} &= (4k^2 x_i x_j - 2k\delta_{ij}) \frac{E}{D}, \quad h_{tt} = (4k^2 \beta^2 t^2 - 2k\beta) \frac{E}{D}. \end{aligned}$$

Hence, for any constant $a \geq 0$,

$$\begin{aligned} \tilde{L}(h - a) &= \frac{E}{D} \left\{ a_{ij}(4k^2 x_i x_j - 2k\delta_{ij}) - b_i(2kx_i) + (4k^2 \beta^2 t^2 - 2k\beta) \right. \\ &\quad \left. + 2\alpha \tan(\alpha t)(2k\beta t) - (\alpha^2 - c) \right\} + (\alpha^2 - c) \frac{e^{-k}}{D} + a(\alpha^2 - c). \end{aligned}$$

Now move h down, and then up to position $h - a$, $a \geq 0$, so that its graph first touches that of v from below, at some point (\bar{x}, \bar{t}) . We claim that $a = 0$, so that $u \geq h$ and we would have the desired conclusion. To see this, suppose $a > 0$, then $(\bar{x}, \bar{t}) \in E_1 \setminus \bar{E}_\delta$. Thus, in view of (24), we have at (\bar{x}, \bar{t}) that

$$\begin{aligned} 2\bar{\epsilon} &\geq \tilde{L}v \geq \tilde{L}(h - a) \geq \frac{E}{D} \left\{ a_{ij}(4k^2 x_i x_j - 2k\delta_{ij}) - b_i(2kx_i) \right. \\ &\quad \left. + (4k^2 \beta^2 t^2 - 2k\beta) + 2\alpha \tan(\alpha t)(2k\beta t) - (\alpha^2 - c) \right\}. \end{aligned}$$

It follows that for some constants $\bar{k}, c_0 > 0$, depending only on n, λ, Λ and δ ,

$$2\bar{\epsilon} \geq c_0 \bar{k}^2 \frac{E}{D}.$$

Then at (\bar{x}, \bar{t}) we have

$$2\bar{\epsilon}(e^{-\bar{k}\delta} - e^{-\bar{k}})e^{\bar{k}(|x|^2 + \beta t^2)} \geq c_0 \bar{k}^2.$$

This shows that if $\bar{\epsilon}$ is small then this is impossible. □

The following example shows that the smallness of $\bar{\epsilon}$ in Lemma 1.2 indeed depends on δ .

In $B_1 \subset \mathbb{R}^2$, consider, for $0 < \delta < 1/4$, the function

$$w(x) = \begin{cases} -\log(|x| + \delta)(1 - \delta - |x|)^2, & \text{for } 0 \leq |x| \leq 1 - \delta, \\ 0, & \text{for } 1 - \delta \leq |x| \leq 1. \end{cases}$$

It is easy to check that $w \in C^2(\overline{B}_1)$, and

$$\Delta w \leq C, \quad \text{for some constant independent of } \delta.$$

Now

$$w(|x|) \geq -\log(2\delta)(1 - 2\delta)^2, \quad |x| \leq \delta.$$

Hence

$$u := w / [-\log(2\delta)(1 - 2\delta)^2]$$

satisfies

$$u \geq 0 \quad \text{in } B_1, \quad u \geq 1 \quad \text{in } B_\delta,$$

and, for some positive constant C' independent of δ ,

$$Lu \leq \frac{C'}{|\log \delta|}.$$

But $u(x) = 0$ for $1 - \delta \leq |x| \leq 1$. Thus in the lemma, it is necessary that $\bar{\epsilon} < C'' / |\log \delta|$ for some C'' smaller than C' .

We will also use

Lemma 1.3 *Let $(a_{ij}(x))$, $b_i(x)$ and $c(x)$ satisfy (13) for some positive constants λ and Λ , $u \in LSC(\Omega)$ satisfy (15), and let ω be a non-negative non-decreasing continuous function*

on $(0, 2d)$, $d := \text{diam}(\Omega)$). Assume that u satisfies, for some $\bar{x}, \bar{y} \in \Omega$, $\bar{x} \neq \bar{y}$, and p, q in \mathbb{R}^n , that

$$u(y) \geq u(\bar{x}) + p \cdot (y - \bar{x}) - |y - \bar{x}| \omega(|y - \bar{x}|), \quad y \in \Omega,$$

and

$$u(z) \geq u(\bar{x}) + p \cdot (\bar{y} - \bar{x}) - |\bar{y} - \bar{x}| \omega(|\bar{y} - \bar{x}|) + q \cdot (z - \bar{y}) - |z - \bar{y}| \omega(|z - \bar{y}|), \quad \forall z \in \Omega.$$

Then, for some positive constants C_1 and C_2 depending only on n, λ, Λ, C ,

$$|p - q| \leq C_1 \omega(2|\bar{x} - \bar{y}|) + C_2 C |\bar{x} - \bar{y}|,$$

where C is the constant in (15).

Proof of Lemma 1.3. Working with $\tilde{u}(z) = u(z + \bar{x}) - [u(\bar{x}) + p \cdot z]$ instead of $u(z)$, we may assume without loss of generality that $\bar{x} = 0$, $u(0) = 0$, $p = 0$, $\bar{y} \neq 0$, $q \neq 0$:

$$u(z) \geq -|z| \omega(|z|), \quad z \in \Omega, \quad (25)$$

$$u(z) \geq -|\bar{y}| \omega(|\bar{y}|) + q \cdot (z - \bar{y}) - |z - \bar{y}| \omega(|z - \bar{y}|), \quad \forall z \in \Omega. \quad (26)$$

By (25),

$$u(z) \geq -2|\bar{y}| \omega(2|\bar{y}|), \quad \forall 0 < |z| \leq 2|\bar{y}|.$$

For $|z - \bar{y}| \leq \frac{1}{2}|\bar{y}|$, we deduce from (26) that

$$u(z) \geq -|\bar{y}| \omega(|\bar{y}|) + q \cdot (z - \bar{y}) - \frac{1}{2}|\bar{y}| \omega\left(\frac{1}{2}|\bar{y}|\right) \geq -2|\bar{y}| \omega(2|\bar{y}|) + q \cdot (z - \bar{y}).$$

It follows that

$$u(z) \geq -2|\bar{y}| \omega(2|\bar{y}|) + \frac{1}{4}|q||\bar{y}|, \quad \forall z \in (B_{2|\bar{y}|}(0) \setminus B_{|\bar{y}|/2}(\bar{y})) \cap U, \quad (27)$$

where

$$U := \{z \in \mathbb{R}^n \mid q \cdot (z - \bar{y}) \geq \frac{1}{2}|q||z - \bar{y}|\}.$$

We may assume that

$$|q| \geq 32\omega(2|\bar{y}|),$$

since otherwise there is nothing to prove. So we deduce from (27) that

$$u(z) \geq \frac{1}{8}|q||\bar{y}| \geq 4|\bar{y}| \omega(2|\bar{y}|), \quad \forall z \in (B_{2|\bar{y}|}(0) \setminus B_{|\bar{y}|/2}(\bar{y})) \cap U,$$

Let

$$\tilde{u}(x) := u(|\bar{y}|x)/(|q||\bar{y}|), \quad x \in B_2.$$

Then we have

$$\begin{aligned} \tilde{u}(0) &= 0, \\ \tilde{u} &\geq -2\omega(2|\bar{y}|)/|q|, \quad \text{in } B_2, \\ \tilde{u} &\geq \frac{1}{8}, \quad (B_2 \setminus B_{1/2}(e)) \cap \tilde{U}, \end{aligned}$$

and

$$\tilde{a}_{ij}\partial_{ij}\tilde{u} + |\bar{y}|\tilde{b}_i\partial_i\tilde{u} + |\bar{y}|^2\tilde{c}\tilde{u} \leq C|\bar{y}|/|q|, \quad \text{in } B_2,$$

where $e = \bar{y}/|\bar{y}|$,

$$\tilde{U} := \{x \in \mathbb{R}^n \mid q \cdot (x - e) \geq \frac{1}{2}|q||x - e|\},$$

and

$$\tilde{a}_{ij}(x) = a_{ij}(|\bar{y}|x), \quad \tilde{b}_i(x) = b_i(|\bar{y}|x), \quad \tilde{c}(x) = c(|\bar{y}|x).$$

Applying Lemma 1.2 to $u(x) = 8[\tilde{u}(2x) + 2\omega(2|\bar{y}|)/|q|]$, we have, for some $\bar{\epsilon} > 0$, depending only on n, λ, Λ ,

$$\frac{\omega(2|\bar{y}|)}{|q|} > \bar{\epsilon}, \quad \text{or} \quad \frac{C|\bar{y}|}{|q|} > \bar{\epsilon}.$$

The desired estimate follows. Lemma 1.3 is established. □

Proof of Proposition 1.1. Given Lemma 1.3, the proof is the same as that of lemma 3.1 in [3] — using this lemma instead of lemma A there. □

The proof of Theorem 1.2 is completed.

1.4 A supersolution outside a closed submanifold

Theorem 1.3 *For $n \geq 1$, let $F \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n})$ satisfy (14), and let $\Omega \subset \mathbb{R}^n$ be a domain, $E \subset \Omega$ be a smooth closed submanifold of dimension k , $0 \leq k \leq n - 1$, and let $f \in USC(\Omega)$. Assume that $u \in LSC(\Omega)$ is lowerconical in E , and satisfies*

$$F(x, u, \nabla u, \nabla^2 u) \leq f(x) \quad \text{in } \Omega \setminus E, \quad \text{in the viscosity sense.} \quad (28)$$

Then

$$F(x, u, \nabla u, \nabla^2 u) \leq f(x) \quad \text{in } \Omega, \quad \text{in the viscosity sense.}$$

Theorem 1.4 For $n \geq 2$, let $F \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n})$ satisfy (14), and let $\Omega \subset \mathbb{R}^n$ be a domain, $E \subset \Omega$ be a smooth closed submanifold of dimension k , $0 \leq k \leq n - 2$, and let $f \in USC(\Omega)$. Assume that $u \in LSC(\Omega \setminus E)$ satisfies

$$\inf_{\Omega \setminus E} u > -\infty, \quad (29)$$

and

$$\lambda_1(\nabla^2 u) + \cdots + \lambda_{k+2}(\nabla^2 u) \leq 0, \quad \text{in } \Omega \setminus E, \text{ in the viscosity sense,} \quad (30)$$

Then, after extending u to E by letting

$$u(x) := \liminf_{y \in \Omega \setminus E, y \rightarrow x} u(y),$$

u is in $LSC(\Omega)$, and is lowerconical in E .

A corollary of the above two theorems is

Corollary 1.1 For $n \geq 2$, let $F \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n})$ satisfy (14), and let $\Omega \subset \mathbb{R}^n$ be a domain, $E \subset \Omega$ be a smooth closed submanifold of dimension k , $0 \leq k \leq n - 2$, and let $f \in USC(\Omega)$. Assume that $u \in LSC(\Omega \setminus E)$ satisfies (29), (30) and

$$F(x, u, \nabla u, \nabla^2 u) \leq f(x) \quad \text{in } \Omega \setminus E, \text{ in the viscosity sense.}$$

Then

$$F(x, u, \nabla u, \nabla^2 u) \leq f(x) \quad \text{in } \Omega, \text{ in the viscosity sense.}$$

Remark 1.3 In the above theorem, condition (30) is only needed to be satisfied, for some $\bar{r} > 0$, in $E_{\bar{r}} \setminus E$, $E_{\bar{r}} = \{x \mid \text{dist}(x, E) < \bar{r}\}$, since we can apply the theorem with $\Omega = E_{\bar{r}}$.

Our proof of Theorem 1.4 makes use of the following maximum principle for functions satisfying (30).

Proposition 1.2 For $n \geq 2$, $-1 \leq k \leq n - 2$, let E be a smooth closed k -dimensional manifold in \mathbb{R}^n and $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $u \in LSC(\overline{\Omega} \setminus E)$ satisfies (30) and

$$\inf_{\Omega \setminus E} u > -\infty.$$

Then

$$u \geq \inf_{\partial\Omega \setminus E} u, \quad \text{on } \Omega \setminus E.$$

Note that in the above, when $k = -1$, E is understood as \emptyset , the empty set; while for $k = 0$, E consists of finitely many points.

Remark 1.4 *The above proposition was proved in [3] under a stronger assumption that $u \in C^2(\Omega \setminus E) \cap C^0(\overline{\Omega} \setminus E)$. The proof applies with minor modification.*

Proof of Theorem 1.3. Let $\varphi \in C^2(\Omega)$ satisfy

$$\varphi \leq u, \quad \text{in } \Omega, \quad \text{and } \varphi(\bar{x}) = u(\bar{x}), \text{ for some } \bar{x} \in \Omega.$$

We only need to prove (17). If $\bar{x} \in \Omega \setminus E$, this follows from (28). We may assume, without loss of generality, that $\bar{x} = 0 \in E$.

For any fixed $\delta > 0$, let

$$\varphi^{(\delta)}(x) := \varphi(x) - \frac{\delta}{2}|x|^2.$$

Consider, for $0 < \epsilon < 1$,

$$\varphi_\epsilon(x) := \varphi^{(\delta)}(x) + \epsilon d(x),$$

where $d(x) := \text{dist}(x, E)$ denotes the distance function from x to E . Since u is lowerconical on E , we have, for small $\epsilon > 0$, that

$$\lambda(\epsilon) := -\inf_{\Omega \setminus E} (u - \varphi_\epsilon) > 0.$$

Since $\varphi_\epsilon = \varphi^{(\delta)}$ on E , and $u - \varphi^{(\delta)} > 0$ on $\overline{\Omega} \setminus \{\bar{x}\}$, we have, for small $\epsilon > 0$,

$$u - \varphi_\epsilon > 0 \text{ on } \partial\Omega, \quad \liminf_{x \in \Omega \setminus E, d(x) \rightarrow 0} (u - \varphi_\epsilon)(x) \geq 0.$$

Therefore, there exists $x_\epsilon \in \Omega \setminus E$ such that

$$u - \tilde{\varphi}_\epsilon \geq 0 \text{ in } \Omega \setminus E, \quad (u - \tilde{\varphi}_\epsilon)(x_\epsilon) = 0,$$

where

$$\tilde{\varphi}_\epsilon(x) := \varphi_\epsilon(x) - \lambda(\epsilon) = \varphi^{(\delta)}(x) + \epsilon d(x) - \lambda(\epsilon).$$

Using the positivity of $u - \varphi^{(\delta)}$ in $\overline{\Omega} \setminus \{0\}$, we obtain from the above that

$$\lambda(\epsilon) = -(u - \varphi_\epsilon)(x_\epsilon) = \varphi^{(\delta)}(x_\epsilon) - u(x_\epsilon) + \epsilon d(x_\epsilon) \leq \epsilon d(x_\epsilon),$$

and

$$\lim_{\epsilon \rightarrow 0} d(x_\epsilon) = \lim_{\epsilon \rightarrow 0} |x_\epsilon| = 0.$$

By (28),

$$F(x_\epsilon, \tilde{\varphi}_\epsilon(x_\epsilon), \nabla \tilde{\varphi}_\epsilon(x_\epsilon), \nabla^2 \tilde{\varphi}_\epsilon(x_\epsilon)) \leq f(x_\epsilon).$$

We know from the above that, as $\epsilon \rightarrow 0$,

$$x_\epsilon \rightarrow 0, \quad \tilde{\varphi}_\epsilon(x_\epsilon) \rightarrow \tilde{\varphi}_\epsilon(0) = \varphi(0), \quad \nabla \tilde{\varphi}_\epsilon(0) \rightarrow \nabla \varphi(0).$$

For

$$\nabla^2 \tilde{\varphi}_\epsilon(x_\epsilon) = \nabla^2 \varphi^{(\delta)}(x_\epsilon) + \epsilon \nabla^2 d(x_\epsilon),$$

we use lemma 7.1 in [3] to obtain that $d(x)$ is pseudoconvex near E , i.e.

$$\nabla^2 d(x_\epsilon) \geq O(1), \quad \text{as } \epsilon \rightarrow 0$$

— this is probably a known result. Thus, using (14), we have

$$\begin{aligned} f(0) &\geq \lim_{\epsilon \rightarrow 0} f(x_\epsilon) \geq \lim_{\epsilon \rightarrow 0} F(x_\epsilon, \tilde{\varphi}_\epsilon(x_\epsilon), \nabla \tilde{\varphi}_\epsilon(x_\epsilon), \nabla^2 \tilde{\varphi}_\epsilon(x_\epsilon)) \\ &\geq \lim_{\epsilon \rightarrow 0} F(x_\epsilon, \tilde{\varphi}_\epsilon(x_\epsilon), \nabla \tilde{\varphi}_\epsilon(x_\epsilon), \nabla^2 \varphi^{(\delta)}(x_\epsilon) + O(\epsilon)) \\ &= F(0, \varphi(0), \nabla \varphi(0), \nabla^2 \varphi(0) + \delta I). \end{aligned}$$

Sending δ to 0 in the above leads to the desired inequality (17). Theorem 1.3 is established. \square

Now we give the

1.5 Proof of Theorem 1.4

It is clear that the extension u is in $LSC(\Omega)$. We will prove that u is lowerconical in E .

Fix any $\bar{x} \in E$ and any $\eta \in C^\infty(\Omega)$, we will show that (4) holds for every $\epsilon > 0$. We prove this by contradiction. Suppose not, then for some $\bar{\epsilon} > 0$,

$$\left((u + \eta)(x) - (u + \eta)(\bar{x}) - \bar{\epsilon} \text{dist}(x, E) \right) \geq 0, \quad \forall x \in \Omega. \quad (31)$$

We may assume $\bar{x} = 0 \in E$, and the tangent space of E at 0 is spanned by e_{n-k+1}, \dots, e_n , where $e_1 = (1, 0, \dots, 0)$, \dots , $e_n = (0, \dots, 0, 1)$ are the standard basis of \mathbb{R}^n . We write $x = (x', x'')$, where $x' = (x_1, \dots, x_{n-k})$ and $x'' = (x_{n-k+1}, \dots, x_n)$.

For x close to 0, we have, for some constant C ,

$$\frac{3}{4}|x'| - C|x''|^2 \leq |x'| - C|x|^2 \leq d(x) \equiv \text{dist}(x, E) \leq |x'| + C|x| \leq \frac{5}{4}|x'| + C|x''|^2. \quad (32)$$

The above fact follows easily from (70) in [3].

It follows from (31) and (32) that for some $\bar{r}, \bar{C} > 0$,

$$\begin{aligned} u(x) &\geq u(0) + \eta(0) - \eta(x) + \bar{\epsilon} \text{dist}(x, E) = u(0) - \nabla \eta(0) \cdot x + O(|x|^2) + \bar{\epsilon} \text{dist}(x, E) \\ &\geq u(0) - \nabla \eta(0) \cdot x + \frac{\bar{\epsilon}}{2}|x'|^2 - \bar{C}|x''|^2, \quad x \in B_{\bar{r}} \setminus E. \end{aligned}$$

For $0 < a < \min\{\bar{r}, \bar{\epsilon}/4\}$ which will be chosen later, let

$$h(x) := u(0) - \nabla \eta(0) \cdot x + \frac{\bar{\epsilon}}{4a}|x'|^2 - (\bar{C} + 1)|x''|^2 + \frac{a^2}{2}.$$

On $\partial B_a(0) \setminus E$,

$$u(x) \geq u(0) - \nabla \eta(0) \cdot x + \frac{\bar{\epsilon}}{2a}|x'|^2 - \bar{C}|x''|^2 = h(x) + \frac{\bar{\epsilon}}{4a}|x'|^2 + |x''|^2 - \frac{a^2}{2} \geq h(x).$$

By lemma 8.2 in Appendix B of [3] and the assumption (30), there exists some positive constant $\bar{a} > 0$ such that if we further require $0 < a < \bar{a}$, we have

$$\sum_{i=1}^{k+2} \lambda_i(\nabla^2(u - h)) \leq 0, \quad \text{in } \Omega \setminus E, \quad \text{in the viscosity sense.} \quad (33)$$

Indeed, let $\varphi \in C^2(\Omega \setminus E)$ satisfy

$$\varphi \leq u - h, \quad \text{in } \Omega \setminus E, \quad \text{and } \varphi(\bar{x}) = (u - h)(\bar{x}), \text{ for some } \bar{x} \in \Omega.$$

Then, by (30),

$$\sum_{i=1}^{k+2} \lambda_i(\nabla^2 h(\bar{x}) + \nabla^2 \varphi(\bar{x})) \leq 0.$$

Applying the above mentioned lemma in [3], with $l = k + 2$,

$$D = \frac{2a}{\bar{\epsilon}} \nabla^2 h(\bar{x}) = \text{diag}(1, \dots, 1, -\delta_1, \dots, -\delta_k), \quad \delta_1 = \dots = \delta_k = \frac{4a(\bar{C} + 1)}{\bar{\epsilon}},$$

$$M = D + \frac{2a}{\bar{\epsilon}} \nabla^2 \varphi(\bar{x}),$$

we obtain

$$\sum_{i=1}^{k+2} \lambda_i(\nabla^2 \varphi(\bar{x})) = \frac{\bar{\epsilon}}{2a} \sum_{i=1}^{k+2} \lambda_i(M - D) \leq \sum_{i=1}^{k+2} \lambda_i(M) = \frac{2a}{\bar{\epsilon}} \sum_{i=1}^{k+2} \lambda_i(\nabla^2 h(\bar{x}) + \nabla^2 \varphi(\bar{x})) \leq 0.$$

We have proved (33). Thus, in view of proposition 1.2 in [3],

$$u - h \geq \inf_{\partial B_a(0) \setminus E} (u - h) \geq 0,$$

and therefore

$$\liminf_{x \rightarrow 0, x \in \Omega \setminus E} u(x) \geq h(0) = u(0) + \frac{a^2}{2} > u(0).$$

A contradiction. We have therefore proved (4). Theorem 1.4 is established. \square

2 Maximum principle

In a bounded open set Ω in \mathbb{R}^n we consider two functions, u, v ; u is in $LSC(\overline{\Omega})$, and $v \in C^2(\Omega) \cap C(\overline{\Omega})$. The function u is assumed to satisfy, in Ω ,

$$F(x, u, \nabla u, \nabla^2 u) \leq F(x, v, \nabla v, \nabla^2 v) \quad \text{in viscosity sense.} \quad (34)$$

Here $F(x, s, p, M)$ is continuous and its derivatives in (s, p, M) are continuous. Concerning ellipticity of F we assume here that F may be degenerate elliptic, but that there is a unit vector ξ , such that for all values of the arguments of F ,

$$F_{M_{ij}} \xi_i \xi_j > 0.$$

However we do not assume that this expression is uniformly bounded by some positive constant.

Theorem 2.1 (*maximum principle*) *Assume*

$$u \geq v \quad \text{on } \partial\Omega, \quad \inf_{\Omega} u > -\infty. \quad (35)$$

Then $u \geq v$ in Ω provided

$$F_s(x, s, p, M) \leq 0, \quad \forall (s, x, p, M). \quad (36)$$

For a uniformly elliptic operator one knows that even if (36) is not assumed, the conclusion $u \geq v$ holds if the volume of Ω is small. However if there is no uniform ellipticity this needs not hold. Here is an

Example 2.1 Let $\Omega = B_R$, $a = R^{-2}$, and let

$$F(x, u, \nabla u, \nabla^2 u) := -e^{-\Delta u} + 1 + u - \frac{1}{2}x \cdot \nabla u.$$

Then

$$u(x) := -1 + a|x|^2, \quad \text{and } v(x) \equiv 0$$

satisfy

$$F(x, u, \nabla u, \nabla^2 u) \leq 0 = F(x, v, \nabla v, \nabla^2 v), \quad \text{in } B_R,$$

and

$$u = v \text{ on } \partial B_R.$$

But

$$u < v \text{ in } B_R.$$

Note that R may be arbitrarily small.

Before proving Theorem 2.1 it is convenient to subtract u , and to consider $F(x, s, p, M) - F(x, v(x), \nabla v(x), \nabla^2 v(x))$ in place of F . Then for the new u and F we have

$$F(x, u, \nabla u, \nabla^2 u) \leq 0 = F(x, 0, 0, 0) \quad \text{in viscosity sense,} \quad (37)$$

and

$$u \geq 0 \quad \text{on } \partial\Omega. \quad (38)$$

From now on we assume u satisfies (37) and (38). Condition (36) continues to hold.

Proof of Theorem 2.1. We may suppose that $F_{M_{11}} > 0$. We argue by contradiction. Assume there is a point, which we take as origin, where u assumes its minimum value $-k$, $k > 0$.

Suppose

$$\min_{\overline{\Omega}} x_1 = -R.$$

We use the comparison function

$$h(x) := -k + ke^{-\lambda(x_1+R)} - e^{-\lambda R}$$

with $\lambda > 0$ to be chosen large. We have

$$h(0) = -k, \quad h < 0 \quad \text{on } \partial\Omega.$$

Move h down, i.e. subtract a constant from h so that it lies below u , then move it up, to value

$$h - c_0, \quad c_0 \geq 0$$

so that its graph first touches that of u at some point \bar{x} . Since $h < 0$ on $\partial\Omega$, $\bar{x} \in \Omega$. We have

$$h_i = -k\lambda\delta_{i1}e^{-\lambda(x_1+R)}, \quad h_{ij} = k\lambda^2\delta_{i1}\delta_{j1}e^{-\lambda(x_1+R)}.$$

Because of (37),

$$F(\bar{x}, h(\bar{x}), \nabla h(\bar{x}), \nabla^2(\bar{x})) \leq 0.$$

Here all the arguments are bounded in absolute value by some constant independent of λ and if we use the theorem of the mean, and the fact that $F(x, 0, 0, 0) = 0$, we see that

$$0 \geq a_{ij}h_{ij} + b_i h_i + c h =: I \quad (39)$$

with (a_{ij}) uniformly positive definite, and all coefficients bounded in absolute value. In addition, by (36),

$$c \leq 0. \quad (40)$$

Computing, we find

$$\begin{aligned} I &= k\lambda^2 a_{11} e^{-\lambda(x_1+R)} - k\lambda b_1 e^{-\lambda(x_1+R)} + c(h - c_0) \\ &\geq k e^{-\lambda(x_1+R)} [a_{11}\lambda^2 - \lambda b_1 + c]; \end{aligned}$$

where we have used (40). But since $a_{11} > 0$, for large λ this is positive, contradicts (39). \square

3 Strong maximum principle and Hopf Lemma for viscosity solutions

We take up first the strong maximum principle. Here

$$u \geq v$$

are functions defined in Ω , an open and connected subset of \mathbb{R}^n , u is in $LSC(\Omega)$ while v is in $C^2(\Omega)$. The function u satisfies

$$F(x, u, \nabla u, \nabla^2 u) \leq F(x, v, \nabla v, \nabla^2 v) \quad \text{in viscosity sense.}$$

The nonlinear operator $F(x, s, p, M)$ is continuous and of class C^1 in (s, p, M) for all values of the arguments. F is assumed to be elliptic, i.e.

$$\left(\frac{\partial F}{\partial M_{ij}}\right) \text{ is positive definite,}$$

for all values of the arguments. However F is not assumed to be uniformly elliptic, nor are $|F_s|$, $|F_{p_i}|$ uniformly bounded.

Theorem 3.1 (*Strong maximum principle*) *Let u and v be as above. Suppose $u = v$ at some point in Ω . Then*

$$u \equiv v.$$

Before giving the proof, it is convenient to change u and F . Namely, if we subtract v from u and $F(x, v(x), \nabla v(x), \nabla^2 v(x))$ from F , we may then assume that

$$u \geq 0$$

and

$$F(x, u, \nabla u, \nabla^2 u) \leq 0 = F(x, 0, 0, 0) \quad \text{in viscosity sense.} \quad (41)$$

From now on we assume that u satisfies (41).

Proof of Theorem 3.1. We argue by contradiction. Suppose the conclusion is false. Since u is LSC, and nonnegative, the set where $u = 0$ is closed. Then there is an open ball B of radius R , with $\overline{B} \subset \Omega$, with $u > 0$ in \overline{B} except that $u(\hat{x}) = 0$ at some point \hat{x} on ∂B ; we may suppose the center is the origin.

As in the classical proof of the strong maximum principle we make use of a comparison function

$$h(x) = E(x) - e^{-\alpha R^2}, \quad E(x) := e^{-\alpha|x|^2}, \quad \alpha > 0 \text{ to be chosen large.} \quad (42)$$

Then

$$h_i = -2\alpha x_i E, \quad h_{ij} = (4\alpha^2 x_i x_j - 2\alpha \delta_{ij}) E.$$

Let A be an open ball, with $\overline{A} \subset \Omega$, having \hat{x} as center, of radius $\delta = \delta(\alpha) < R/2$ satisfying

$$\delta \alpha^{1/2} < \frac{\pi}{10}. \quad (43)$$

Clearly

$$-1 \leq h \leq 1, \quad |\nabla h| + |\nabla^2 h| \leq C, \quad \text{in } A, \quad (44)$$

where C is some constant independent of α .

Now, in the ball \overline{A} we change u and F to \tilde{u} and \tilde{F} to ensure that

$$\tilde{F}_u < 0$$

for values of the arguments bounded, say, by 1: We set

$$u = \tilde{u} \xi, \quad \xi := \frac{1}{\alpha} \cos((x_1 - \hat{x}_1) \alpha^{1/2}) \quad (45)$$

By (43),

$$\cos((x_1 - \hat{x}_1)\alpha^{1/2}) > \frac{1}{2} \quad \text{in } \bar{A}.$$

Then set

$$\tilde{F}(x, \tilde{u}, \nabla \tilde{u}, \nabla^2 \tilde{u}) := F(x, \tilde{u}\xi, \nabla(\tilde{u}\xi), \nabla^2(\tilde{u}\xi)),$$

so that \tilde{u} satisfies

$$\tilde{F}(x, \tilde{u}, \nabla \tilde{u}, \nabla^2 \tilde{u}) \leq 0 \quad \text{in viscosity sense.} \quad (46)$$

For some $\bar{\epsilon} = \bar{\epsilon}(\alpha) > 0$, we have

$$\tilde{u} \geq \epsilon h, \quad \text{on } \partial A, \quad \forall 0 < \epsilon < \bar{\epsilon}.$$

Now move ϵh down, i.e. subtract a constant from it so that it lies below \tilde{u} in \bar{A} . Then move it up, it becomes

$$\epsilon h - c_0, \quad 0 \leq c_0 = c_0(\epsilon, \alpha) \leq \epsilon,$$

so that its graph first touches that of \tilde{u} at some point \bar{x} . Then, because of (46), we have

$$F(\bar{x}, (\epsilon h - c_0)\xi, \nabla((\epsilon h - c_0)\xi)(\bar{x}), \nabla^2((\epsilon h - c_0)\xi)(\bar{x})) \leq 0.$$

Because of (44), we can fix some constant $\epsilon \in (0, \bar{\epsilon}(\alpha))$ such that

$$\max\{|(\epsilon h - c_0)(\bar{x})|, |\nabla(\epsilon h - c_0)(\bar{x})|, |\nabla^2(\epsilon h - c_0)(\bar{x})|\} \leq 1.$$

Thus, at $(\bar{x}, (\epsilon h - c_0)(\bar{x}), \nabla(\epsilon h - c_0)(\bar{x}), \nabla^2(\epsilon h - c_0)(\bar{x}))$, $F_{M_{ij}}$ is uniformly positive definite and $|F_{p_i}|$, $|F_s|$ are bounded independent of α . By the theorem of the mean we find that at \bar{x} ,

$$F_{M_{ij}}(\epsilon h - c_0)_{ij} + [2F_{M_{ij}}\xi_j\xi^{-1} + F_{p_i}](\epsilon h - c_0)_i + c(\epsilon h - c_0) \leq 0. \quad (47)$$

Claim. $c < 0$.

Proof. Here

$$c\xi = F_{M_{ij}}\xi_{ij} + F_{p_i}\xi_i + F_s\xi,$$

where the arguments in F and its derivatives are all bounded, independent of α . Also $F_{M_{ij}}$ is uniformly positive definite and $|F_{p_i}|$, $|F_s|$ are bounded independent of α . Hence, for large α ,

$$c\xi = -F_{M_{11}} \cos[(x_1 - \hat{x}_1)\alpha^{1/2}] + O(\alpha^{-1/2}) < 0.$$

Since $c < 0$ we see that

$$c(\epsilon h - c_0) = c\epsilon E - c(\epsilon e^{-\alpha k^2} + c_0) > c\epsilon E.$$

Inserting this in (47) we infer that

$$F_{M_{ij}}h_{ij} + [2F_{M_{ij}}\xi_j\xi^{-1} + F_{p_i}]h_i + cE < 0,$$

i.e.

$$F_{M_{ij}}(4\alpha^2x_ix_j - 2\alpha\delta_{ij})E + [2F_{M_{i1}}\xi_1\xi^{-1} + F_{p_i}](-2\alpha x_iE) + cE < 0.$$

Since $|x|$ is bounded away from zero in \overline{A} , we see from the above that for some positive constant c_1, c_2 independent of α ,

$$(c_1\alpha^2 - c_2\alpha - c_2\alpha^{3/2} - c_2\alpha - c_2)E < 0.$$

But this is impossible for α large.

□

Next

Theorem 3.2 (*Hopf Lemma*) *Let u and v be as above, with $u > v$ in Ω , and suppose*

$$u(\hat{x}) = v(\hat{x})$$

at a boundary point \hat{x} near which $\partial\Omega$ is C^2 . Then, if ν is the unit interior normal to $\partial\Omega$ at \hat{x} ,

$$\liminf_{s \rightarrow 0^+} \frac{(u - v)(\hat{x} + s\nu)}{s} > 0.$$

Proof. As before, by considering $u - v$ in place of u , etc. we may suppose

$$u > 0 \quad \text{in } \Omega, \quad u(\hat{x}) = 0,$$

$$F(x, u, \nabla u, \nabla^2 u) \leq 0 \quad \text{in viscosity sense in } \Omega.$$

Let B be a ball of radius R , in Ω with \hat{x} on its boundary. We take the origin as center of B . We use the same comparison function h as in (42). As in the proof of Theorem 3.1, let A be an open ball with center at \hat{x} , with radius $\delta(\alpha) < R/2$ satisfying (43). We work in the region

$$D = B \cap A.$$

In \overline{B} we introduce as before the function \tilde{u} defined as in (45), and \tilde{F} .

Again, for small $\epsilon > 0$ we have

$$\tilde{u} > \epsilon h \quad \text{on } \partial D. \tag{48}$$

Now, argue as in the proof of Theorem 3.1. Move ϵh down and then up so that it becomes

$$\epsilon h - c_0, \quad 0 \leq c_0 = c_0(\epsilon, \alpha) \leq \epsilon,$$

and so that its graph first touches that of \tilde{u} from below at some point \bar{x} .

Claim. $c_0 = 0$.

If that is the case, then $\tilde{u} \geq \epsilon h$ in \overline{D} and the desired conclusion,

$$\liminf_{s \rightarrow 0^+} \frac{u(\hat{x} + s\nu)}{s} > 0,$$

follows.

Proof of Claim. Suppose not, suppose $c_0 > 0$. Because of (48), \bar{x} is in D . Then arguing exactly as in the proof of Theorem 3.1 we are led to a contradiction. □

At the end of this section we point out that Theorem 2.1 can be deduced from Theorem 3.1 as follows.

Theorem 2.1 as a consequence of Theorem 3.1. Suppose the contrary, then

$$\inf_{\Omega} (u - v) < 0. \tag{49}$$

Move v down so that its graph lies below that of u , then move it up to value

$$v - c_0, \quad c_0 \geq 0,$$

so that its graph first touches that of u at some point $\bar{x} \in \overline{\Omega}$. Namely,

$$u \geq v - c_0 \quad \text{in } \Omega, \quad u(\bar{x}) = v(\bar{x}) - c_0.$$

By (35) and (49), $c_0 > 0$ and $\bar{x} \in \Omega$. By (36), we have

$$F(x, v, \nabla v, \nabla^2 v) \leq F(x, v - c_0, \nabla(v - c_0), \nabla^2(v - c_0)).$$

Thus, in view of (34),

$$F(x, u, \nabla u, \nabla^2 u) \leq F(x, v - c_0, \nabla(v - c_0), \nabla^2(v - c_0)) \quad \text{in } \Omega, \text{ in viscosity sense.}$$

Applying Theorem 3.1 to u and $v - c_0$, we infer that $u = v - c_0$ in the connected component of Ω containing \bar{x} . This violates $u \geq v$ on $\partial\Omega$. □

4 Strong maximum principle and Hopf Lemma for viscosity solution of nonlinear parabolic equation

In this section we extend the strong maximum principle, Theorem 3.1, to nonlinear parabolic operators.

In the closure $\overline{\Omega}$ of a domain Ω in \mathbb{R}^{n+1} , (x, t) space, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, we consider two functions

$$u \geq v,$$

u is lower semicontinuous (LSC) while $v \in C^2$; u satisfies, in the viscosity sense, in Ω ,

$$F(x, t, u, \nabla u, \nabla^2 u) - u_t \leq F(x, t, v, \nabla v, \nabla^2 v) - v_t. \quad (50)$$

Here ∇ and ∇^2 represent first and second derivatives with respect to the x -variables. $F(x, t, s, p, M)$ is as in section 3: F is continuous and of class C^1 in (s, p, M) for all values of the arguments. F is assumed to be elliptic, i.e.

$$\left(\frac{\partial F}{\partial M_{ij}}\right) \text{ is positive definite,}$$

for all values of the arguments. However F is not assumed to be uniformly elliptic, nor are $|F_s|$, $|F_{p_i}|$ uniformly bounded.

Setup. We assume that Ω lies in $\{t < T\}$ for some T and that $\partial\Omega$ includes a relatively open subset Σ on the hypersurface $\{t = T\}$.

For every point $P = (x_0, t_0) \in \Omega \cup \Sigma$, we denote by C_P the arcwise connected component, containing P , of points (x, t_0) , in $\Omega \cup \Sigma$. We emphasize that $C_P \subset \{t = t_0\}$.

We also denote by S_P the set of points in Ω which may be connected to P by a continuous curve on which the t -coordinate is nondecreasing.

We require also that

$$(50) \text{ holds not only on } \Omega, \text{ but also at points of } \Sigma. \quad (51)$$

Remark 4.1 *This makes sense. It would not make sense if we require (50) to hold at point (\bar{x}, \bar{t}) on a lower boundary point of Ω , i.e. when Ω lies in $\{t > \bar{t}\}$.*

The main result of this section, the parabolic strong maximum principle, is

Theorem 4.1 *Let Ω , and $u \geq v$ be as above. If $u(P) = v(P)$ at a point on Σ then*

$$u \equiv v \quad \text{in } C_P \cup S_P.$$

Before starting the proof it is convenient, as in the elliptic case, to change u and F . If we subtract v from u , and $F(x, t, v, \nabla v, \nabla^2 v)$ from F then we may suppose

$$u \geq 0 \equiv v, \quad \text{in } \Omega \cup \Sigma, \quad (52)$$

$$F(x, t, u, \nabla u, \nabla^2 u) - u_t \leq 0 = F(x, t, 0, 0, 0), \quad \text{in } \Omega \cup \Sigma, \text{ in the viscosity sense.} \quad (53)$$

From now on we assume u satisfies (52) and (53).

The proof follows that of the parabolic strong maximum principle in Nirenberg [4], with modifications for viscosity supersolution. It makes use of several lemmas.

Lemma 4.1 *Consider u satisfying (52) and (53); suppose also $u > 0$ in a ball, B , with $\overline{B} \subset \Omega$,*

$$u = 0 \quad \text{at a point } P \text{ on } \partial B.$$

Then necessarily, the vector $(0, \dots, 0, 1)$ is normal to ∂B at P .

We postpone the proof; first make some corollaries:

Corollary 4.1 *If $u > 0$ in a subdomain G of Ω , with $\overline{G} \subset \Omega$ and $u(P) = 0$ at a point P on ∂G where ∂G is smooth, then $(0, \dots, 0, 1)$ is normal to ∂G at P .*

This follows from Lemma 4.1 by just taking a ball B in G with P on its boundary.

Corollary 4.2 *Let u satisfy (52) and (53). If $u(P) = 0$ for some point $P \in \Omega$ then*

$$u \equiv 0 \quad \text{on } C_P.$$

Proof. Suppose not, suppose $u(Q) > 0$ for some $Q \in C_P$. Join Q to P by a continuous curve on C_P . As we traverse the curve from Q to P , let \bar{P} be the first point where $u = 0$; it may be P . Let \bar{Q} be a point on the curve so close to \bar{P} that $B_{8|\bar{Q}-\bar{P}|}(\bar{Q}) \subset \Omega$.

Since u is LSC there is a small vertical segment, i.e. parallel to t -axis, of length 2ϵ , $0 < \epsilon < |\bar{Q} - \bar{P}|$, with center at \bar{Q} , where $u > 0$. If $\bar{Q} = (\bar{x}, \bar{t})$, the closed ellipsoid

$$E_a := \{(x, t) \mid (t - \bar{t})^2 + a^{-2}|x - \bar{x}|^2 \leq \epsilon^2\}$$

lies in Ω , provided $0 < a\epsilon < |\bar{Q} - \bar{P}|$.

For small $a > 0$, $u > 0$ in E_a . Now increase a , as we do so, u remains positive in E_a . This follows from Corollary 4.1. Finally, $u > 0$ in E_a for $a = \bar{a} := |\bar{Q} - \bar{P}|/\epsilon$. But \bar{P} lies on the boundary of $E_{\bar{a}}$. Contradiction.

□

We will also use

Lemma 4.2 *Let u satisfy (52) and (53). Suppose $u > 0$ in open half ball D in $t < T_0 \leq T$ centered at (\bar{x}, T_0) . Then $u > 0$ also on the relatively open part of the flat boundary of D , where $t = T_0$.*

Before proving the lemmas we first show how they give the

Proof of Theorem 4.1. Suppose it does not hold, i.e. there is some point Q in $C_P \cup S_P$ where $u(Q) > 0$. Without loss of generality, since u is LSC, we may suppose that Q is in S_P . Join Q to P by a continuous curve Γ on which t is nondecreasing. As we traverse the curve from Q , let $P_0 = (x_0, t_0)$ be the first point where $u = 0$ (it may be P). Let D be an open half ball with center at P_0 whose closure lies in $(\Sigma \cup \Omega) \cap \{t \leq t_0\}$. Near the end of Γ it lies in D . By Corollary 4.2, $u > 0$ in D . Then, by Lemma 4.2, $u(P_0) > 0$. Contradiction. Theorem 4.1 is proved. \square

We now prove the lemmas. First

Proof of Lemma 4.2. Suppose the conclusion is false, we suppose $u(P) = 0$ at a point $P = (x_0, t_0)$ on the relatively open part of the flat boundary of D . We may take x_0 to be the origin. In addition, for convenience, we take $T_0 = 0$. So $P = \{0\}$.

Near the origin we introduce the comparison function

$$h = -\alpha t - |x|^2, \quad 0 < \alpha \text{ to be chosen large.}$$

In the cutoff region

$$D_\alpha := \{(x, t) \mid -\alpha^{-2} < t < 0, \alpha t + |x|^2 < 0\},$$

we have

$$|x| < \frac{1}{\sqrt{\alpha}}, \quad 0 < h < \frac{1}{\alpha}, \quad h_t = -\alpha, \quad h_i = -2x_i, \quad h_{ij} = -2\delta_{ij}. \quad (54)$$

Since $u > 0$ on the boundary of D_α where also $t = -\alpha^{-2}$, $u > \epsilon h$ there for small $\epsilon > 0$ (the smallness of ϵ may depend on α). Thus, on ∂D_α , $u \geq \epsilon h$, with equality only at $\{0\}$.

Now, move h down, i.e., subtract a positive constant from h , so that it lies below \tilde{u} in \overline{D}_δ . Then move it up to $\epsilon h - c_0$, $0 \leq c_0 \leq \epsilon/\alpha$, so that its graph first touches that of u at some point $(\bar{x}, \bar{t}) \in D_\alpha \cup \{0\}$.

Since u satisfies (51), at (\bar{x}, \bar{t}) ,

$$F(x, t, \epsilon h - c_0, \epsilon \nabla h, \epsilon \nabla^2 h) - \epsilon h_t \leq 0.$$

We will show that this cannot hold for α large. For $0 < \epsilon$ small, the arguments $(\epsilon h - c_0, \epsilon \nabla h, \epsilon \nabla^2 h)$ are all bounded in absolute value by 1. Hence, by the ellipticity of F and the fact that $F(x, t, 0, 0, 0) = 0$,

$$0 \geq -\epsilon h_t + \epsilon a_{ij} h_{ij} + \epsilon b_i h_i + c(\epsilon h - c_0) =: J,$$

with (a_{ij}) uniformly positive definite, and all coefficients bounded in absolute value.

We now compute J . Using (54) and the inequalities

$$h \leq \frac{1}{\alpha}, \quad 0 \leq c_0 \leq \epsilon/\alpha,$$

we find for a fixed constant C independent of α that

$$\frac{J}{\epsilon} \geq \alpha - C - \frac{C}{\alpha} > 0 \quad \text{for } \alpha \text{ large.}$$

Contradiction. □

Now,

Proof of Lemma 4.1. By taking a smaller ball inside B with P on its boundary, we may suppose that

$$u > 0 \text{ on } \overline{B} \text{ except at } P.$$

As usual we argue by contradiction. Suppose the conclusion is false. We may suppose that the origin is the center of B , its radius is R , and $P = (\hat{x}, \hat{t})$, $\hat{x} \neq 0$. We use the comparison function

$$h = E - e^{-\alpha R^2}, \quad E = e^{-\alpha(|x|^2 + t^2)}.$$

We have

$$h_t = -2\alpha t E, \quad h_i = -2\alpha x_i E, \quad h_{ij} = (-2\alpha \delta_{ij} + 4\alpha^2 x_i x_j) E,$$

Let A be a small ball centered at P , with radius $\delta < |\hat{x}|/2$ and $\overline{A} \subset \Omega$. We require $\delta = \delta(\alpha)$ to be small, namely we require

$$\delta \alpha^{1/2} < \frac{\pi}{10},$$

so that

$$\cos[(x_1 - \hat{x}_1)\alpha^{1/2}] > \frac{1}{2} \quad \text{in } \overline{A}.$$

Now, in the ball \overline{A} we change u and F to \tilde{u} and \tilde{F} , to ensure that

$$\tilde{F}_u < 0$$

for values of the arguments bounded, say, by 1. Namely, we set

$$u = \tilde{u}\xi, \quad \xi = \frac{\cos[(x_1 - \hat{x}_1)\alpha^{1/2}]}{\alpha}.$$

Then we set

$$\tilde{F}(x, t, \tilde{u}, \nabla \tilde{u}, \nabla^2 \tilde{u}) = \frac{1}{\xi} F(x, t, \tilde{u}\xi, \nabla(\tilde{u}\xi), \nabla^2(\tilde{u}\xi)),$$

so \tilde{u} satisfies

$$\tilde{F}(x, t, \tilde{u}, \nabla \tilde{u}, \nabla^2 \tilde{u}) - \tilde{u}_t \leq 0 \quad \text{in viscosity sense.}$$

For some $\bar{\epsilon} = \bar{\epsilon}(\alpha) > 0$, we have

$$\tilde{u} \geq \epsilon h \quad \text{on } \partial A, \quad \forall 0 < \epsilon < \bar{\epsilon}.$$

As we did in the proof of Lemma 4.2, move ϵh down so that it lies below \tilde{u} , and then move it up, so that it becomes

$$\epsilon h - c_0, \quad 0 \leq c_0 = c_0(\epsilon, \alpha) \leq \epsilon,$$

so that its graph first touches that of \tilde{u} at some point (\bar{x}, \bar{t}) . Then at (\bar{x}, \bar{t}) ,

$$I := F(\bar{x}, \bar{t}, (\epsilon h - c_0)\xi, \nabla((\epsilon h - c_0)\xi), \nabla^2((\epsilon h - c_0)\xi)) - \epsilon \xi h_t \leq 0.$$

By the theorem of the mean we find that, at (\bar{x}, \bar{t}) ,

$$F_{M_{ij}}(\epsilon h - c_0)_{ij} + [2F_{M_{ij}}\xi_j\xi^{-1} + F_{p_i}](\epsilon h - c_0)_i + c(\epsilon h - c_0) - \epsilon h_t \leq 0,$$

where

$$c\xi = F_{M_{ij}}\xi_{ij} + F_{p_i}\xi_i + F_s\xi,$$

and the arguments in F and its derivatives are all bounded independent of α ; also, $F_{u_{ij}}$ is uniformly positive definite and $|F_{p_i}|, |F_s|$ are bounded.

We claim that $c < 0$.

For large α ,

$$c\xi = -F_{M_{11}} \cos[(x_1 - \hat{x}_1)\alpha^{1/2}] + O(\alpha^{-1/2}) < 0, \quad \text{for } \alpha \text{ large.}$$

Using the fact that $c < 0$, we argue as in the proof of Theorem 3.1 to obtain

$$F_{M_{ij}}(4\alpha^2 x_i x_j - 2\alpha \delta_{ij})E + [2F_{M_{i1}}\xi_1\xi^{-1} + F_{p_i}](-2\alpha x_i E) + cE - h_t < 0.$$

Since $|x|$ is bounded away from zero in \overline{A} , we see from the above that for some positive constant c_1, c_2 independent of α ,

$$(c_1\alpha^2 - c_2\alpha - c_2\alpha^{3/2} - c_2\alpha - c_2 - c_2\alpha)E < 0.$$

But this is impossible for α large. Contradiction. Lemma 4.1 is proved. \square

Using similar arguments we now prove a parabolic Hopf Lemma for viscosity supersolutions.

Consider Ω and Σ , and u, v as above, with

$$u > v \quad \text{in } \Omega \cup \Sigma.$$

We will prove the parabolic Hopf Lemma at a point, which we take to be the origin $\{0\}$, on $\partial\Sigma$.

$\overline{\Omega} \setminus \Sigma$ is called the parabolic boundary, $P\partial\Omega$ of Ω , and we assume that it is of class C^2 near $\{0\}$. For convenience we suppose that $\nu = (0, \dots, 0, 1, 0)$ is the inner normal to $\partial\Sigma$ (of class C^2) at $(0, 0)$, and we denote x_n by y . Sometimes we use (x, y, t) with $x = (x_1, \dots, x_{n-1})$. We assume that the interior normal to $P\partial\Omega$ at $\{0\}$ is not $(0, \dots, 0, 1)$.

We now suppose $u > v$ in $\Omega \cup \Sigma$ and $u(0, 0) = v(0, 0)$, and we assume (51), i.e.

$$F(x, t, u, \nabla u, \nabla^2 u) - u_t \leq F(x, t, v, \nabla v, \nabla^2 v) - v_t \quad \text{in } \Omega \cup \Sigma \quad \text{in viscosity sense.}$$

Theorem 4.2 (*Parabolic Hopf Lemma*) *Under the conditions above,*

$$\liminf_{s \rightarrow 0^+} \frac{(u - v)(s\nu)}{s} > 0. \tag{55}$$

Remark 4.2 *It will be clear from the proof that (55) will also hold for any unit vector $\nu = (\nu_1, \dots, \nu_{n+1})$ at $\{0\}$ which points into $\Omega \cup \Sigma$ and is not tangent to $P\partial\Omega$, so, $\nu_{n+1} \leq 0$.*

As before, by considering $u - v$ in place of u , and subtracting $F(x, t, v, \nabla v, \nabla^2 v)$ from F we may assume $v \equiv 0$ and

$$F(x, t, u, \nabla u, \nabla^2 u) - u_t \leq 0 = F(x, t, 0, 0, 0) \quad \text{in } \Omega \cup \Sigma, \quad \text{in viscosity sense.}$$

Proof of Theorem 4.2. By restricting Ω we may assume that near $\{0\}$, $\partial\Sigma$ is given by

$$y = d|x|^2, \quad d > 0,$$

and that for some constant $b > 0$, the domain

$$\widehat{\Omega} = \{(x, y, t) \mid t < 0, y > d|x|^2 - bt\},$$

near the origin, lies in Ω . By decreasing d and increasing b we may suppose that for the resulting $\hat{\Omega}$, which we now call Ω ,

$$u > 0 \quad \text{on } P\partial\Omega \text{ except at } \{0\}. \quad (56)$$

We will take b to be large.

Next we introduce the comparison function

$$h = y - d|x|^2 - bt.$$

With

$$A = \text{ball centered at origin with radius } \delta \text{ small,}$$

we consider u and h in the region

$$G = \Omega \cap A.$$

Since (56) holds we see that for some $0 < \epsilon$ small,

$$u > \epsilon h \quad \text{on } P\partial G.$$

The desired conclusion (55),

$$\liminf_{s \rightarrow 0^+} \frac{u(s\nu)}{s} > 0,$$

will follow if we can show that $u \geq \epsilon h$ on \overline{G} .

To achieve this we argue as before: lower ϵh so that it lies below u in \overline{G} and then raise it to

$$\epsilon h - c_0,$$

until its graph first touches that of u . We claim this must happen for $c_0 = 0$, which would prove

$$u \geq \epsilon h.$$

Suppose not, suppose $c_0 > 0$ and that the point of contact is (\bar{x}, \bar{t}) . Clearly (\bar{x}, \bar{t}) is not on $P\partial\Omega$; \bar{t} might be zero. At (\bar{x}, \bar{t}) we have

$$F(x, t, h, \nabla h, \nabla^2 h) - \epsilon h_t \leq 0.$$

All arguments in F are bounded by 1, for ϵ small, so that we may infer, as before, that

$$0 \geq a_{ij}h_{ij} + b_i h_i + ch - h_t.$$

With the operator on the right uniformly elliptic and with coefficients uniformly bounded. Thus for some C independent of b ,

$$0 \geq -C + b > 0 \quad \text{for } b \text{ large.}$$

Contradiction. Then $c_0 = 0$, i.e.

$$u \geq \epsilon h.$$

□

5 A strengthened Hopf Lemma for viscosity solution of parabolic equations

In this section we extend Lemma 1.1 to parabolic equations. The result is not used in this paper. On the other hand it is useful when extending Theorem 1.2 to parabolic equations. We plan to extend Theorem 1.1-1.4 to parabolic equations in a forthcoming paper.

Let $\Omega \subset \mathbb{R}^n$ be a domain with C^2 boundary, $0 < T < \infty$. Assume that $(a_{ij}(x, t))$, $b_i(x, t)$ and $c(x, t)$ are functions in $L^\infty(\Omega \times (0, T])$ satisfying, for some positive constants λ and Λ ,

$$|a_{ij}(x, t)| + |b_i(x, t)| + |c(x, t)| \leq \Lambda, \quad a_{ij}(x, t)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega, \quad 0 < t < T, \quad \xi \in \mathbb{R}^n. \quad (57)$$

We will use the notation

$$Lu := a_{ij}(x, t)\partial_{ij}u + b_i(x, t)\partial_i u + c(x, t)u.$$

Theorem 5.1 *For $0 < T_1 < T < \infty$ and $0 < \delta < 1$, let $(a_{ij}(x, t))$, $b_i(x, t)$ and $c(x, t)$ be $L^\infty(\Omega \times (0, T])$ functions satisfying (57) with $\Omega = B_1$ for some positive constants λ and Λ . There exist some positive constants $\epsilon, \mu > 0$ which depend only on $n, \lambda, \Lambda, \delta, T_1, T$, such that if $u \in LSC(\overline{B_1} \times (0, T])$ satisfies*

$$(L - \partial_t)u \leq \epsilon, \quad \text{in } B_1 \times (0, T], \quad \text{in viscosity sense}, \quad (58)$$

$$u(x, 0) \geq 1, \quad \text{for } |x| \leq \delta, \quad (59)$$

$$u \geq 0, \quad \text{on } P\partial(B_1 \times (0, T]). \quad (60)$$

Then

$$u(x, t) \geq \mu(1 - |x|), \quad \text{on } B_1 \times [T_1, T]. \quad (61)$$

Recall that $P\partial(B_1 \times (0, T])$ denotes the parabolic boundary of $B_1 \times (0, T]$, i.e.

$$P\partial(B_1 \times (0, T]) = (\overline{B_1} \times \{0\}) \cup (\partial B_1 \times [0, T]).$$

Note. The function u may actually be negative somewhere.

Theorem 5.2 *Let Ω be a domain of \mathbb{R}^n with C^2 boundary, and let $B \subset \Omega$ be a ball. For $0 < T_1 < T < \infty$, let $(a_{ij}(x, t))$, $b_i(x, t)$ and $c(x, t)$ be $L^\infty(\Omega \times (0, T])$ functions satisfying (57) for some positive constants λ and Λ . There exist some positive constants $\epsilon, \mu > 0$ which depend only on $n, \lambda, \Lambda, \Omega$, the radius of B , T_1, T , such that if $u \in LSC(\overline{\Omega} \times (0, T])$ satisfies*

$$\begin{aligned} (L - \partial_t)u &\leq \epsilon, \quad \text{in } \Omega \times (0, T], \text{ in viscosity sense,} \\ u(x, 0) &\geq 1 \quad \text{for all } x \in B, \\ u &\geq 0 \quad \text{on } P\partial(\Omega \times (0, T]). \end{aligned} \tag{62}$$

Then

$$u(x, t) \geq \mu \text{dist}(x, \partial\Omega), \quad \text{on } \Omega \times [T_1, T].$$

Proof of Theorem 5.1. We only need to prove that there exists some constant \overline{T} depending only on $n, \lambda, \Lambda, \delta$ such that the theorem holds under an additional assumption that $T \leq \overline{T}$. Indeed, for general T , we fix a positive integer m so that $\frac{T}{m} < \overline{T}$, and then apply the result on $[0, \frac{T}{m}]$, $[\frac{T}{m}, \frac{2T}{m}]$, ..., $[\frac{(m-1)T}{m}, T]$ successively. We leave the simple details to readers.

In the following we will assume that $T \leq \overline{T}$, and we will determine the value of \overline{T} later.

We may assume without loss of generality that $c(x, t) \leq 0$ for all $|x| < 1$ and $0 < t < T$. This can be achieved by working with

$$\tilde{u}(x, t) = e^{-2\Lambda t} u(x, t),$$

since (58) implies

$$(\tilde{L} - \partial_t)\tilde{u} := (L - 2\Lambda - \partial_t)\tilde{u} = e^{-2\Lambda t}(L - \partial_t)u \leq \epsilon,$$

and \tilde{L} has $\tilde{c} = c - 2\Lambda \leq -\Lambda < 0$.

Consider the comparison function

$$h(x, t) := \frac{1}{D}(E - F), \quad E = (t + a)^{-k} e^{-\frac{\alpha|x|^2}{t+a}}, \quad F = (T + a)^{-k} e^{-\frac{\alpha}{T+a}},$$

with

$$\alpha = \frac{1}{\lambda},$$

and

$$\frac{T + a}{4a} = \frac{2}{\delta^2}.$$

Thus

$$a := (8\delta^{-2} - 1)^{-1}T. \quad (63)$$

Next we require that

$$h(\frac{\delta}{2}, 0) = 0, \quad (64)$$

i.e.

$$k := \frac{\alpha}{(T+a)\log(1+\frac{T}{a})} = \frac{\alpha(8-\delta^2)}{8\log(8\delta^{-2})} \frac{1}{T}. \quad (65)$$

Clearly,

$$k(T+a) = \frac{\alpha}{\log(8\delta^{-2})} < \alpha. \quad (66)$$

Next we choose D so that

$$h(0, 0) = 1, \quad (67)$$

i.e.

$$D = a^{-k} - F.$$

Since

$$h(x, T) = 0, \quad \text{for } |x| = 1,$$

and, in view of (66),

$$D\partial_t h(x, t) = \frac{E}{(t+a)^2} [\alpha - k(t+a)] > 0, \quad \text{for } |x| = 1, \quad 0 \leq t \leq T,$$

we have

$$h(x, t) \leq 0, \quad \text{for } |x| = 1, \quad 0 \leq t \leq T. \quad (68)$$

We see from (64), (67) and the expression of h , that

$$h(x, 0) \leq h(\frac{\delta}{2}, 0) = 0, \quad \text{for } |x| \geq \frac{\delta}{2},$$

and

$$h(x, 0) \leq h(0, 0) = 1, \quad \text{for all } x.$$

Thus, in view of (59) and (60),

$$h \leq u \quad \text{on } P\partial(B_1 \times (0, T]). \quad (69)$$

Claim. There exists constants $\overline{T} > 0$, which depends only on $n, \lambda, \Lambda, \delta$, such that for all $0 < T < \overline{T}$,

$$(L - \partial_t)h \geq \epsilon, \quad \text{in } B_1 \times (0, T], \quad (70)$$

where $\epsilon > 0$ is some constant depending only on $n, \lambda, \Lambda, \delta$, and T .

Proof of the Claim. We compute

$$h_i = -\frac{2\alpha x_i}{t+a} \frac{E}{D}, \quad h_{ij} = \left(\frac{4\alpha^2 x_i x_j}{(t+a)^2} - \frac{2\alpha \delta_{ij}}{t+a} \right) \frac{E}{D},$$

$$-h_t = \left(\frac{k}{t+a} - \frac{\alpha |x|^2}{(t+a)^2} \right) \frac{E}{D}.$$

Thus

$$J := (t+a) \frac{D}{E} (Lh - h_t) = \frac{a_{ij} 4\alpha^2 x_i x_j}{t+a} - 2\alpha \sum_i a_{ii} - 2\alpha b_i x_i + c(t+a)$$

$$-c \frac{F}{E} (t+a) + k - \frac{\alpha |x|^2}{t+a}.$$

By our choice of $\alpha = 1/\lambda$, and also $|x| \leq 1$, and $c < 0$, we have, for some constant C depending only on n, λ, Λ and δ ,

$$J \geq k - C(1+T) = \frac{(8-\delta^2)}{8\lambda \log(8\delta^{-2})} \frac{1}{T} - C(1+T).$$

Clearly, there exists some constant $\bar{T} > 0$ which depends only on $n, \lambda, \Lambda, \delta$, such that for all $0 < T < \bar{T}$, we have

$$J \geq \frac{(8-\delta^2)}{9\lambda \log(8\delta^{-2})} \frac{1}{T}.$$

On the other hand

$$(t+a) \frac{D}{E} \leq (T+a) \frac{a^{-k}}{E} \leq (T+a)^{k+1} a^{-k} e^{\frac{|x|^2}{\lambda(t+a)}} \leq (T+a)^{k+1} a^{-k} e^{\frac{1}{\lambda a}}.$$

The claim follows immediately from the above.

Let $\bar{T} > 0$ be the positive constant in the above claim, and assume that $T \in (0, \bar{T})$. Let $\epsilon > 0$ be the constant in the claim, which depends on T in particular, and let u satisfy the hypotheses of Theorem 5.1 with this ϵ . We will show that

$$u \geq h \quad \text{in } B_1 \times (0, T]. \quad (71)$$

This implies

$$u(x, T) \geq \mu(1 - |x|), \quad \forall x \in B_1,$$

where $\mu > 0$ is some constant depending only on $n, \lambda, \Lambda, \delta$ and T .

Since the positive constants ϵ and μ can clearly be chosen to depend on T monotonically, the above implies (61).

Now we prove (71): Lower the graph of h to be below that of u , in $B_1 \times [0, T]$, and then move it up to a position

$$h - c_0,$$

so that its graph touches that of u from below at some point (\bar{x}, \bar{t}) . It suffices to prove that $c_0 \leq 0$. Suppose not, $c_0 > 0$. Then,

$$|\bar{x}| < 1, \quad 0 < \bar{t} \leq T.$$

Since $Lu - u_t \leq \epsilon$ in viscosity sense, we have, at (\bar{x}, \bar{t}) ,

$$L(h - c_0) - h_t \leq \epsilon.$$

It follows, since $c < 0$,

$$(L - \partial_t)h(\bar{x}, \bar{t}) \leq \epsilon + cc_0 < \epsilon.$$

This contradicts (70). We have proved (71). Theorem 5.1 is established. □

Proof of Theorem 5.2. As usual, we always assume, without loss of generality, that $c(x, t) \leq 0$ on $(0, T]$.

If the assumption (62) is replaced by

$$u \geq 0 \quad \text{in } \Omega \times (0, T],$$

then the conclusion can be deduced from Theorem 5.1 by using arguments similar to that used in the proof of Lemma 1.1.

Since

$$(L - \partial_t)(u + \epsilon t) = 0, \quad \text{in } \Omega \times (0, T],$$

and

$$u + \epsilon t \geq u \geq 0, \quad \text{on } P\partial(\Omega \times (0, T]),$$

we have

$$u + \epsilon t \geq 0, \quad \Omega \times (0, T].$$

Thus, as mentioned above, the conclusion of Theorem 5.2 holds for $u + \epsilon t$. Namely, for some positive constant $\bar{\mu}$ depending only on $n, \lambda, \Lambda, \Omega$, the radii of B , T_1, T , but independent of ϵ , such that

$$u + \epsilon t \geq \bar{\mu} \operatorname{dist}(x, \partial\Omega), \quad \text{on } \Omega \times [T_1/2, T]. \quad (72)$$

Let $d(x) = \text{dist}(x, \partial\Omega)$ denote the distance of x to $\partial\Omega$, and we work in $\Omega \setminus \overline{\Omega}_\delta$, where

$$\Omega_\delta := \{x \in \Omega \mid \text{dist}(x) > \delta\}$$

for small δ . The value of δ , depending only on Ω , will be fixed below.

For $0 < \epsilon \leq \frac{\bar{\mu}\delta}{2T}$, we see from (72) that

$$u \geq \bar{\mu}\delta - \epsilon T \geq \frac{\bar{\mu}\delta}{2}, \quad \text{on } \partial\Omega_\delta \times [T_1/2, T], \quad (73)$$

and

$$u \geq -\epsilon T_1/2 + \bar{\mu}d(x), \quad \text{on } (\Omega \setminus \overline{\Omega}_\delta) \times \{T_1/2\}. \quad (74)$$

Fix a function $\rho \in C^\infty([T_1/2, \infty))$ satisfying $\rho(t) = 0, t \geq T_1$; $-T_1/2 \leq \rho(t) \leq 0, T_1/2 \leq t \leq T_1$; $\rho(T_1/2) = -T_1/2$; $-2 \leq \rho'(t) \leq 0, t \geq 0$. We use comparison

$$h(x, t) := \frac{\bar{\mu}}{4} \left(d(x) + \frac{d(x)^2}{2\delta} \right) + \epsilon \rho(t).$$

A computation shows (see e.g. lemma 7.1 in [3]) that for some small positive numbers δ and a , depending only on Ω , we have

$$L \left(d(x) + \frac{d(x)^2}{2\delta} \right) \geq \frac{a\lambda}{\delta}, \quad \text{in } (\Omega \setminus \overline{\Omega}_\delta).$$

Thus, after further requiring that $\epsilon < \frac{a\lambda\bar{\mu}}{12\delta}$, we have

$$(L - \partial_t)h \geq \frac{a\lambda\bar{\mu}}{4\delta} - 2\epsilon \geq \epsilon, \quad \text{in } (\Omega \setminus \overline{\Omega}_\delta) \times (0, T].$$

Now we have

$$(L - \partial_t)u \leq (L - \partial_t)h, \quad \text{in } (\Omega \setminus \overline{\Omega}_\delta) \times [T_1/2, T]$$

and

$$u \geq h, \quad \text{on } P\partial(\Omega \setminus \overline{\Omega}_\delta) \times [T_1/2, T],$$

it follows that

$$u \geq h, \quad \text{in } (\Omega \setminus \overline{\Omega}_\delta) \times [T_1/2, T].$$

In particular

$$u \geq h = \frac{\bar{\mu}}{2} \left(d(x) + \frac{d(x)^2}{2\delta} \right), \quad \text{in } (\Omega \setminus \overline{\Omega}_\delta) \times [T_1, T].$$

Theorem 5.2 is established. □

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